Elementary Number Theory - Exam Practice Problems

1. Prove that the equation $x^2 + 2y^2 = 8z + 5$ has no integer solutions.
2. Find all positive integers $x < 300$ such that
$$\begin{aligned} x &\equiv 3 \mod 4 \\ x &\equiv 4 \mod 5 \\ x &\equiv 2 \mod 7. \end{aligned}$$
3. Find all incongruent solutions of the congruence $x^3 \equiv 25 \mod 64$.
4. Find all incongruent solutions of the congruence $x^2 + 3x + 18 \equiv 0 \mod 28$.
5. Show that if p is a prime with $p \equiv 1 \mod 3$ then $x^2 + x + 1 \equiv 0 \mod p$ has a solution.
6. A *Carmichael number* is an odd composite number n which satisfies $a^{n-1} \equiv 1 \mod n$ for every a with $\gcd(a, n) = 1$. Show that $561 = 3 \cdot 11 \cdot 17$ is a Carmichael number.
7. Show that if p is a prime and $0 < k < p$ then $(p-k)!(k-1)! \equiv (-1)^k \mod p$.
8. Let $n > 4$ be an integer. Show that $n \mid (n-1)!$ if and only if n is composite.
9. Let $f(n)$ be an arithmetic function such that $\Sigma_{d|n} f(d) = n^2 + 1$. Compute $f(12)$. Is f multiplicative?
10. Recall that $\tau(n)$ denotes the number of positive divisors of n. Let $k \geq 2$ be a fixed integer. Show that the equation $\tau(n) = k$ has infinitely many solutions.
11. Given that 5 is a primitive root modulo 73, find all positive integers less than 73 having order 12 modulo 73.
12. Find all primitive roots α modulo $250 = 2 \cdot 5^3$ such that $\alpha \equiv 2, 3 \mod 5$.
13. Let p be the prime $p = 131 = 2 \cdot 5 \cdot 13 + 1$. Use the fact that 53 has order 5 modulo p and that 39 has order 13 to find a primitive root modulo p.
14. Show that if $29a^2 \equiv b^2 \mod 19$ then $29a^2 \equiv b^2 \mod 19^2$.
15. Let p be an odd prime. Show $\left(\frac{-3}{p}\right) = 1 \iff p \equiv 1 \mod 3$.
16. Evaluate the Legendre symbol $\left(\frac{103}{229}\right)$.
17. If n is odd, evaluate the Jacobi symbol $\left(\frac{n^3}{n-2}\right)$. **Hint:** The answer can be written in the form
$$\left(\frac{n^3}{n-2}\right) = \begin{cases} 1 & \text{if} \quad n \equiv u_1, u_2, \ldots, u_s \mod m \\ -1 & \text{if} \quad n \equiv v_1, v_2, \ldots, v_t \mod m \\ 0 & \text{if} \quad n \equiv r_1, r_2, \ldots, r_k \mod m \end{cases}$$
18. If α and α' are two irrational numbers with simple continued fraction expansions $\alpha = [3; 1, 5, 9, a_1, a_2, \ldots]$ and $\alpha' = [3; 1, 5, 7, b_1, b_2, \ldots]$, prove that $|\alpha - \alpha'| < \frac{49}{7095} = \frac{49}{3 \cdot 43 \cdot 55}$.
19. Find the real number with the periodic simple continued fraction expansion $[3; 4, \overline{1, 2}]$.
20. Express $\sqrt{28}$ as a simple continued fraction, and find the solution $(x, y) \in \mathbb{Z}^2$ of the equation $x^2 - 28y^2 = 1$ with the smallest positive y value.